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Hamilton–Jacobi Equations and Nonlinear Control Problems

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1. Introduction and Summary of Results

We study the Hamilton–Jacobi equation

$$\begin{aligned} &\frac{\partial\varphi}{\partial t}(t,x)+h^*(-B^*D_x\varphi(t,x)) \\ &\qquad +(Ax+Fx,D_x\varphi(t,x))=g(x), \qquad x\in H, \quad t\geqslant 0 \\ &\qquad\qquad \varphi(0,x)=\varphi_0(x) \end{aligned} \tag{1.1}$$

in a real Hilbert space H, in connection with the optimal control problem

$$\inf\left\{\int_0^T (g(y(t))+h(u(t)))\,dt+\varphi_0(y(T));\, u\in L^2(0,T;U)\right\} \tag{1.2}$$

where

$$\begin{aligned} y'(t)+Ay(t)+Fy(t)&=Bu(t) \qquad \text{for} \quad t\in[0,T] \\ y(0)&=x. \end{aligned} \tag{1.3}$$

GENERAL ASSUMPTIONS. 1. H, U are real Hilbert spaces with scalar products $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$, respectively. The norms of H and U are denoted $|\cdot|$ and $|\cdot|_U$, respectively.

2. A is a densely defined, closed linear operator on H with the domain denoted $D(A)$; $-A$ is the infinitesimal generator of an analytic C_0-semigroup e^{-At}. For every $t>0$, e^{-At} is compact. B is a linear continuous operator from U to H and B^* denotes its adjoint.

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3. The operator $F: H \to H$ is Fréchet differentiable and admits a locally Lipschitz Fréchet derivative $F': H \to L(H, H)$. There exist some real constants C_1, C_2 such that

$$(Ay + Fy, y) \geqslant -C_1 |y|^2 - C_2 \qquad \forall y \in D(A). \tag{1.4}$$

4. The functions g, $\varphi_0: H \to R$ are Fréchet differentiable and admit locally Lipschitz derivatives ∇g and $\nabla \varphi_0$. There exist α, $\alpha_0 \in H$ and β, $\beta_0 \in R$ such that

$$g(y) \geqslant (\alpha, y) + \beta, \qquad \forall y \in H \tag{1.5}$$

$$\varphi_0(y) \geqslant (\alpha_0, y) + \beta_0, \qquad \forall y \in H. \tag{1.6}$$

5. The function $h: H \to \bar{R} =]-\infty, +\infty]$ is convex, lower semicontinuous and satisfies the growth condition

$$h(u) \geqslant \omega |u|_U^2 + \gamma \qquad \forall u \in U \tag{1.7}$$

for some $\omega > 0$ and $\gamma \in R$.

Denote by $h^*: U \to R$ the conjugate function, i.e.,

$$h^*(p) = \sup\{\langle p, u \rangle - h(u); u \in U\}. \tag{1.7}$$

Assumption 5 implies that h^* is convex and continuous on U. We will further assume that

6. The function h^* is Gâteaux differentiable with locally Lipschitz Gâteaux derivative ∇h^*.

Under our assumtions on A, F for every $u \in L^2(0, T; U)$ and $x \in D(A)$ the Cauchy problem (1.3) has a unique absolutely continuous solution y with $y' = (d/dt) y \in L^2(0, T; H)$. Moreover, the map $u \to y$ is continuous and compact from $L^2(0, T; U)$ to $C([0, T]; H)$. In particular, this implies by standard arguments ([4](that problem (1.2) admits at least one optimal solution $(y, u) \in C([0, T]; H) \times L^2(0, T; U)$.

Define the function $\varphi: R^+ \times H \to R$,

$$\varphi(t, x) = \inf\left\{\int_0^t (g(y(s)) + h(u(s)))\, ds + \varphi_0(y(t));\right.$$

$$\left. y' + Ay + Fy = Bu,\ y(0) = x;\ u \in L^2(0, T; U)\right\} \tag{1.8}$$

and denote by $D_x^- \varphi(t, x)$ the subdifferential of $\varphi(t, \cdot)$ at x, i.e., the set of all $w \in H$ such that (see [6, 9])

$$\liminf_{z \to x} ((\varphi(t, z) - \varphi(t, x) - (w, z - x)) |z - x|^{-1}) \geqslant 0. \tag{1.9}$$

In general, the subset $D_x^- \varphi(t, x)$ is convex and closed for every $x \in H$ (if not empty) and if $x \to \varphi(t, x)$ is convex and lower semicontinuous then $D_x^- \varphi$ coincides with the subdifferential of $x \to \varphi(t, x)$ in the sense of convex analysis. There is also an obvious relation between the subdifferential $D_x^- \varphi(t, x)$ and Clarke's generalized gradient $\partial_x \varphi(t, x)$. More precisely, if $x \to \varphi(t, x)$ is locally Lipschitz then $D_x^- \varphi(t, x) \subset \partial_x \varphi(t, x)$, $\forall x \in H$.

The main result is

THEOREM 1. *Let the general assumptions* 1–6 *be satisfied. Then the function* φ: $[0, T] \times H \to R$ *is continuous, locally Lipschitz in* x *for every* $t \in [0, T]$, *absolutely continuous in* t *for every* $x \in D(A)$ *and* $D_x^- \varphi(t, x) \neq \varnothing$ *for all* $(t, x) \in [0, T] \times H$.

Moreover,

$$\frac{\partial \varphi}{\partial t}(t, x) + h^*(-B^*\eta) + (Ax + Fx, \eta) = g(x) \qquad \text{a.e.} \quad t \in [0, T]$$

$$\text{for all } x \in D(A) \quad \text{and} \quad \text{some } \eta \in D_x^- \varphi(t, x). \tag{1.10}$$

$$\varphi(0, x) = \varphi_0(x), \qquad \forall x \in H. \tag{1.11}$$

If φ_0 *is such that,* $\nabla \varphi_0(p) \in D(A^*)$ *for all* $p \in H$ *then*

$$\frac{\partial \varphi}{\partial t}(t, x) + h^*(-B^*\eta) + (Ax + Fx, \eta) = g(x) \qquad \text{a.e.} \quad t \in [0, T]$$

$$\text{for all } x \in D(A) \qquad \text{and all} \quad \eta \in D_x^- \varphi(t, x). \tag{1.12}$$

For convenience we do not put the above theorem in the most general form.

Without entering into details we note that Theorem 1 can be extended to more general equations of the form

$$\frac{\partial \varphi}{\partial t} + H(t, x, -B^* D_x \varphi) + (Ax + Fx, D_x \varphi) = 0 \tag{1.11}$$

where

$$H(t, x, p) = \sup\{\langle p, u\rangle - L(t, x, u); u \in U\}$$

and L: $[0, T] \times H \times U \to R$ is a certain normal integrand convex with respect to u. One might expect to obtain similar results for more general hamiltonian functions H by associating to Eq. (1.11) an appropriate differential game as in [7].

Theorem 1 represents a sharpening of certain results established in [2–4] eventually under more general conditions.

We shall see later (Proposition 1) that the function φ is a *viscosity solution* to Eq. (1.1) in the sense of Crandall and Lions [5] (see also [6]). Recently these authors utilized this concept to prove existence and uniqueness for a large class of Hamilton–Jacobi equations of the form [7, 8]

$$\frac{\partial \varphi}{\partial t} + H_0(t, x, \varphi, D_x \varphi) = 0 \qquad \text{in} \quad [0, T] \times H \tag{1.12}$$

in a general Banach space X with the Radon–Nykodim property. Although these results do not explicit treat equations of the form (1.1), likely the arguments in [7, 8] work in the present situation. Thus the main interest of Theorem 1 consists in regularity properties of viscosity solutions which under our assumptions are closer to the classical concepts.

Our next result concerns optimal feedback controls for problem (1.2).

THEOREM 2. *Let the assumptions* 1–6 *be satisfied. Then every optimal control u^* to problem* (1.2) *is expressed as a function of corresponding optimal state y^* by the feedback law*

$$u^*(t) \in \nabla h^*(-B^* D_y^- \varphi(T-t, y^*(t))) \qquad \forall t \in [0, T]. \tag{1.13}$$

Equation (1.13) should be inerpreted in the following precise sense:

$$\begin{aligned} u^*(t) = \nabla h^*(-B^*\eta(t)) \qquad &\text{for all } t \in [0, T] \\ &\text{and some } \eta(t) \in D_y^- \varphi(T-t, y^*(t)). \end{aligned} \tag{1.14}$$

Control problems of the form (1.2), (1.3) arise very often as smooth approximations of optimal control problems governed by nonlinear evolution equations

$$\begin{aligned} y' + Ay + \partial\psi(y) \ni Bu \qquad &\text{in} \quad [0, T] \\ y(0) = y_0 \end{aligned} \tag{1.15}$$

where $\partial\psi \colon H \to 2^H$ is the subdifferential of a lower semicontinuous convex function $\psi \colon H \to \bar{R}$. Indeed we may approximate Eq. (1.15) by

$$\begin{aligned} y' + Ay + \nabla\psi^\varepsilon(y) = Bu \qquad &\text{in} \quad [0, T] \\ y(0) = y_0 \end{aligned} \tag{1.16}$$

where ψ^ε is a smooth approximation of ψ (see [1]). In this way we may obtain optimal controls of problem (1.2), (1.15) as weak limits for $\varepsilon \to 0$ of feedback optimal controls of the form (1.13) corresponding to problem (1.2), (1.16). We note that (1.15) represent the general setting for parabolic variational inequalities and free boundary problems.

2. Proof of Theorems 1 and 2

As noted in Section 1 for every $t \in [0, T]$ and $x \in H$ the infimum defining $\varphi(t, x)$ is attained; there is $(y^t, u^t) \in C([0, t]; H) \times L^2(0, t; U)$ such that

$$\varphi(t, x) = \int_0^t (g(y^t(s)) + h(u^t(s)))\, ds + \varphi_0(y^t(t)) \tag{2.1}$$

$$(y^t)' + Ay^t + Fy^t = Bu \quad \text{in} \quad (0, t); y^t(0) = x. \tag{2.2}$$

Then according to the maximum principle, there is $p^t \in C([0, t]; H)$ such that

$$(p^t)' - A^* p^t - (F'(y^t))^* p^t = g(y^t) \quad \text{in} \quad [0, t] \tag{2.3}$$

$$\begin{aligned} p^t(t) &= -\nabla\varphi_0(y^t(t)) \\ u^t(s) &= \nabla h^*(B^* p^t(s)) \qquad \forall s \in [0, t]. \end{aligned} \tag{2.4}$$

The solutions to Eqs. (2.2) and (2.3) are considered in "mild" sense. However, if $x \in D(A)$ then $(y^t)' \in L^2(0, t; H)$ and y is Hölder continuous on $[0, t]$. Hence the function $s \to g(y^t(s))$ is Hölder continuous and by Theorem 7.1 in [10, p. 168] we infer that p^t is a classical solution to Eq. (2.3) on $[0, t)$. Hence u^t is Hölder continuous on every $[0, \delta] \subset [0, t)$ and y^t is a classical solution to (2.2).

We will prove first the following lemma:

Lemma 1. *The function φ is continuous on $[0, T] \times H$. For every $t \in [0, T]$, $\varphi(t, \cdot)$ is locally Lipschitz and for every $x \in D(A)$ the function $t \to \varphi(t, x)$ is Lipschitz continuous on $[0, T]$.*

Proof. The proof is identical with that of Lemma 8 in [3, p. 121] but we outline it for convenience. Denote by $y(s, x, u)$ the solution to Eq. (1.3) on $[0, t]$. Since F is Lipschitz on bounded subset we get by assumption 3^0 the estimates

$$|y(s, x, u)|^2 \leqslant C\left(|x|^2 + \int_0^s |u(\tau)|_U^2\, d\tau\right) \quad \text{for} \quad 0 \leqslant s \leqslant y \tag{2.5}$$

$$|y(s, x, u) - y(s, x, u)| \leqslant C_r\, |x - \bar{x}|, \qquad 0 \leqslant s \leqslant t \tag{2.6}$$

$$\text{if} \quad \|u\|_{L^2(0,t;U)} \leqslant r.$$

where r is arbitrary. On the other hand, for a fixed $u_0 \in U$ we have for $|x| \leqslant r$

$$\int_0^t (g(y^t(s)) + h(u^t(s)))\, ds + \varphi_0(y^t(t))$$

$$\leqslant \int_0^t (g(y(s, x, u_0)) + h(u_0))\, ds + \varphi_0(y(t, x, u_0)) \leqslant C_r^1.$$

Along with assumptions (1.5)–(1.7) the latter implies that for every $r > 0$ there exists $\omega_r > 0$ such that

$$\int_0^t |u^t(s)|_U^2\, ds \leqslant \omega_r \qquad \text{for} \quad |x| \leqslant r. \tag{2.7}$$

Then by (2.6) and the definition of $\varphi(t, x)$ we conclude that for every $r > 0$, $\exists L_r$ such that

$$|\varphi(t, x) - \varphi(t, \tilde{x})| \leqslant L_r\, |x - \tilde{x}| \qquad \text{if} \quad |x|, |\tilde{x}| \leqslant r. \tag{2.8}$$

as claimed.

To prove the second part of Lemma 1 we note first that for all $0 \leqslant s \leqslant t$,

$$\varphi(t, x) = \inf \left\{ \int_0^s (g(y(\tau)) + h(u(\tau)))\, d\tau + \varphi(t - s, y(s)); \right.$$

$$\left. y' + Ay + Fy = Bu \qquad \text{in} \quad [0, s]; y(0) = x \right\}$$

$$= \int_0^s (g(y^t(\tau)) + h(u^t(\tau)))\, d\tau + \varphi(t - s, y^t(s)). \tag{2.9}$$

The latter follows immediately by definition of φ (see Lemma 9 in [3, p. 122]. In particular, we see by (2.8) that

$$|\varphi(t, x) - \varphi(t - s, x)| \leqslant |\varphi(t - s, y^t(s)) - \varphi(t - s, x)|$$

$$+ \int_0^s (|g(y^t)| + |h(u^t)|)\, d\tau. \tag{2.10}$$

On the other hand, it follows by (2.4) and the conjugacy relation (1.7) that

$$h(u^t(s)) = -h^*(B^*p^t(s)) + \langle \nabla h^*(B^*p^t(s)), B^*p^t(s) \rangle.$$

Since by (2.2), (2.3), (2.7), $|p^t(s)| \leqslant C$ for $s \in [0, t]$ where C is independent of t we infer by Assumption 6 that

$$|h(u^t(s))| \leqslant C \qquad \text{for all } 0 \leqslant s \leqslant t \leqslant T. \tag{2.11}$$

Next we see by Eq. (2.2) that

$$|y^t(s)-x| \leqslant |e^{-As}x-x| + \left|\int_0^s e^{-A(s-\tau)}(Bu^t(\tau)-Fy^t(\tau))\,d\tau\right|$$
$$\leqslant s\,|Ax| + Ms, \qquad 0 \leqslant s \leqslant t \leqslant T$$

because as already noted $|u^t(s)| = |\nabla h^*(B^*p^t(s))| \leqslant C$.

Thus from (2.8), (2.10), and (2.11)

$$|\varphi(t,x)-\varphi(t-s,x)| \leqslant L_0 s \qquad \text{for all } s \in [0,t]$$

where L_0 is independent of t. This completes the proof of Lemma 1.

LEMMA 2. *For every $t \in [0,T]$ there exists δ_t such that*

$$(Ay^t(s)+Fy^t(s),p^t(s)) - h^*(B^*p^t(s))$$
$$+\ g(y^t(s)) = \delta_t \qquad \text{for} \quad s \in [0,t]. \tag{2.12}$$

Proof. Forming the scalar product of (2.2) with $(p^t)'$ and of (2.3) with $(y^t)'$, subtracting the results, and using the obvious formulas

$$(Bu^t(s),p^t)'(s)) = (h^*(Bp^t(s)))'; \qquad (\nabla g(y^t),(y^t)') = (g(y^t))'$$

we get (2.12) as claimed.

For every $x \in H$ we set

$$\Gamma x = \{-p^t(0);\, p^t \text{ satisfies Eqs. (2.2)–(2.4)}$$
$$\text{along with some optimal pair } (y^t,u^t)\}. \tag{2.13}$$

As noted earlier $\Gamma x \neq \varnothing$ for all $x \in H$.

LEMMA 3. *For all $x \in H$ and $t \in [0,T]$ we have*

$$\Gamma x = D_x^- \varphi(t,x). \tag{2.14}$$

Proof. We will prove first that

$$\Gamma x \subset D_x^- \varphi(t,x) \qquad \forall (t,x) \in [0,T] \times H.$$

To this end we fix (t,x_0) and consider any solution p^t to Eq. (2.3) corresponding to some optimal pair (y^t,u^t). For any $z \in H$ we have

$$\varphi(t,x_0)-\varphi(t,x) \leqslant \int_0^t (g(\tilde{y}_x)-g(y_x))\,ds$$
$$+\ \varphi_0(\tilde{y}_x(t)) - \varphi_0(y_x(t)) \tag{2.15}$$

where

$$\tilde{\varphi}(t, x) = \int_0^t (g(y_x) + h(u_x))\, ds + \int_0^t |u_x - u^t|_U^2\, ds + \varphi_0(y_x(t))$$

$$= \inf \left\{ \int_0^t (g(y) + h(u))\, ds + \int_0^t |u - u^t|_U^2\, ds \right.$$

$$\left. + \; \varphi_0(y(t));\, y' + Ay + Fy = Bu,\, y(0) = x \right\}$$

and

$$\begin{aligned} y_x' + Ay_x + Fy_x &= Bu_x \qquad \text{in} \quad [0, t] \\ y_x(0) &= x \end{aligned} \tag{2.16}$$

$$\begin{aligned} \tilde{y}_x' + A\tilde{y}_x + F\tilde{y}_x &= Bu_x \qquad \text{in} \quad [0, t] \\ \tilde{y}_x(0) &= x_0. \end{aligned} \tag{2.17}$$

By Eqs. (2.16), (2.17) we see that

$$|y_x(t) - \tilde{y}_x(t)| \leqslant C\, |x_0 - x| \qquad \text{for} \quad t \in [0, T]. \tag{2.18}$$

On the other hand, we have

$$\int_0^t (g(y_x) + h(u_x))\, ds + \int_0^t |u_x - u^t|_U^2\, ds + \varphi_0(y_x(T))$$

$$\leqslant \int_0^t (g(y_x^1) + h(u^t))\, ds + \varphi_0(y_x^1(t)) \tag{2.19}$$

where

$$(y_x^1)' + Ay_x^1 + Fy_x^1 = Bu^t \qquad \text{in} \quad [0, t];\, y_x^1(0) = x_0.$$

Thus letting x tend to x_0 we see by (2.19) that

$$\varphi(t, x_0) + \limsup_{x \to x_0} \int_0^t |u_x - u^t|_U^2\, ds \leqslant \varphi(t, x_0)$$

because

$$\liminf_{x \to x_0} \int_0^t (g(y_x) + h(u_x))\, ds + \varphi_0(y_x(T)) \geqslant \varphi(t, x_0).$$

Hence for $x \to x_0$

$$u_x \to u^t \qquad \text{strongly in } L^2(0, t; U) \tag{2.20}$$

$$y_x, \tilde{y}_x \to y^t \qquad \text{strongly in } C([0, t]; H). \tag{2.21}$$

Then by (2.15) it follows that

$$\begin{aligned}\varphi(t, x_0) - \varphi(t, x) \leqslant \int_0^t (\nabla g(y^t(s)), \tilde{y}_x - y_x)\, ds \\ + (\nabla \varphi_0(y^t(t)), \tilde{y}_x(t) - y_x(t)) \\ + \varepsilon\, |x_0 - x| \qquad \text{for} \quad |x_0 - x| \leqslant \delta(\varepsilon).\end{aligned} \tag{2.22}$$

Let w be the solution to the equation

$$\begin{aligned} w' + Aw + F'(y^t)\, w = 0 \qquad \text{in} \quad [0, t] \\ w(0) = x_0 - x. \end{aligned} \tag{2.23}$$

It is easy to see that if $|x_0 - x| \leqslant \delta(\varepsilon)$ then

$$|\tilde{y}_x(s) - y_x(s) - w(s)| \leqslant \varepsilon\, |x_0 - x| \qquad \forall s \in [0, t].$$

Thus by (2.22) we get

$$\begin{aligned}\varphi(t, x_0) - \varphi(t, x) \leqslant \int_0^t (\nabla g(y^t), w)\, ds + (\nabla \varphi_0(y^t(t)), w(t)) \\ + \varepsilon\, |x_0 - x| \qquad \text{for} \quad |x_0 - x| \leqslant \delta(\varepsilon).\end{aligned} \tag{2.24}$$

Now we take the scalar product of Eq. (2.3) with w and integrate on $[0, t]$ to get (without loss of generality we may assume that $x \in D(A)$)

$$\begin{aligned}\varphi(t, x_0) - \varphi(t, x) \leqslant -(p^t(0), w(0)) + \varepsilon\, |x_0 - x| \\ \text{for} \quad |x_0 - x| \leqslant \delta(\varepsilon).\end{aligned}$$

Hence $-p^t(0) \in D_x^- \varphi(t, x_0)$ as claimed.

Now let $y_0 \in D_x^- \varphi(t, x_0)$ be arbitrary but fixed. According to definition of $D_x^- \varphi$ for every $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that

$$x_0 = \arg\min\{\varphi(t, x) - (y_0, x) + \varepsilon\, |x - x_0|;\ |x - x_0| \leqslant \delta(\varepsilon)\}. \tag{2.24}$$

Define the function ψ_ε,

$$\begin{aligned}\psi_\varepsilon(x) &= \varepsilon\, |x - x_0| && \text{for} \quad |x - x_0| \leqslant \delta(\varepsilon) \\ &= \varepsilon\, |x - x_0|^2/\eta(\varepsilon) && \text{for} \quad |x - x_0| > \delta(\varepsilon).\end{aligned}$$

where $\eta(\varepsilon) > 0$ is sufficiently small.

Then by (2.24) we see that for every $\varepsilon > 0$,

$$\begin{aligned}\varphi(t, x_0) - (y_0, x) + \psi_\varepsilon(x_0) \\ = \inf\{\varphi(t, x) - (y_0, x) + \psi_\varepsilon(x); x \in H\}.\end{aligned} \tag{2.25}$$

Consider the optimization problem

$$\begin{aligned}\inf\Big\{\int_0^t (g(y(s)) + h(u(s)))\, ds + \varphi_0(y(t)) \\ - (y_0, y(0)) + \psi_\varepsilon(y(0)); \\ (y, u) \text{ satisfy (1.8) on } (0, t)\Big\},\end{aligned} \tag{2.26}$$

and note that the infimum is attained in such a pair $(y_\varepsilon, u_\varepsilon)$ satisfying system (1.8) and $y(0) = x_0$. Indeed, we have

$$\begin{aligned}&\inf_{(y,u)}\Big\{\int_0^t (g(y) + h(u))\, ds + \varphi_0(y(t)) - (y_0, y(0)) + \psi_\varepsilon(y(0))\Big\} \\ &= \inf_{x \in H}\Big\{\inf\Big\{\int_0^t (g(y) + h(u))\, ds + \varphi_0(y(t)); y(0) = x, \\ &\quad (y, u) \text{ satisfy (1.3)}\Big\} - (y_0, x) + \psi_\varepsilon(x)\Big\} \\ &= \inf\{\varphi(t, x) - (y_0, x) + \psi_\varepsilon(x); x \in H\}.\end{aligned}$$

Now applying the maximum principle in problem (2.26) we infer that there exists $p_\varepsilon \in C([0, t]; H)$ such that

$$\begin{aligned}p_\varepsilon' - A^* p_\varepsilon - (F'(y_\varepsilon))^* p_\varepsilon = \nabla g(y_\varepsilon) \quad &\text{in} \quad [0, t] \\ p_\varepsilon(0) + y_0 \in \partial\psi_\varepsilon(y_\varepsilon(0)) = \varepsilon \operatorname{sgn} 0 \quad & \qquad\qquad (2.27) \\ p_\varepsilon(t) = -\nabla\varphi_0(y_\varepsilon(t)) \quad & \\ u_\varepsilon = \nabla h^*(B^* p_\varepsilon) \quad &\text{in} \quad [0, t] \qquad (2.28)\end{aligned}$$

where $\operatorname{sgn} 0 = \{x \in H; |x| \leqslant 1\}$.

On the other hand, we have

$$\begin{aligned}y_\varepsilon' + Ay_\varepsilon + Fy_\varepsilon = Bu_\varepsilon \quad &\text{in} \quad [0, t] \\ y_\varepsilon(0) = x_0. &\end{aligned} \tag{2.29}$$

By assumptions (1.5)–(1.7) and the obvious inequality

$$\int_0^t (g(y_\varepsilon)+h(u_\varepsilon))\,ds+\varphi_0(y_\varepsilon(t))-(y_0,x_0)$$

$$\leqslant \int_0^t (g(y)+h(u))\,ds+\varphi_0(y(t))-(y_0,x_0)$$

for all (y,u) satisfying (1.3) and $y(0)=x_0$, we deduce that $\{u_\varepsilon\}$ is weakly compact in $L^2(0,t;U)$ and $\{y_\varepsilon\}$ is compact in $C([0,t];H)$. Thus letting ε tend to zero (on a subsequence) we conclude that

$$u_\varepsilon \to u^t \qquad \text{weakly in } L^2(0,t;U)$$

$$y_\varepsilon \to y^t \qquad \text{strongly in } C([0,t];H)$$

where (y^t,u^t) satisfies Eq. (2.2), $y^t(0)=x_0$ and

$$\varphi(t,x_0)=\int_0^t (g(y^t)+h(u^t))\,ds+\varphi_0(y^t(t)).$$

Similarly, we see by (2.27) that for $\varepsilon\to 0$

$$p_\varepsilon \to p^t \qquad \text{strongly in } C([0,t];H)$$

where p^t satisfies Eqs. (2.3), (2.4) and $p^t(0)=-y_0$. Hence $y_0\in \Gamma x_0$ and the proof of Lemma 3 is complete.

Proof of Theorem 1. We will assume first that $\nabla\varphi_0(y)\in D(A^*)$ for all $y\in H$. Then by Eq. (2.3) we see that p^t is continuously differentiable on $[0,t]$ and therefore by (2.4) we infer that u^t is Lipschitzian on $[0,t]$. Then by Theorem 3.5 in [10] we conclude that y^t is a continuously differentiable solution to Eq. (2.2). Let $x\in D(A)$ be fixed and let $t\in[0,T]$ be such that $\varphi(\cdot,x)$ is differentiable at t. By definition of φ we have

$$\varphi(t-\varepsilon,x)-\varphi(t,x)\leqslant -\int_{t-\varepsilon}^t (g(y^t(s))+h(u^t(s)))\,ds$$
$$+\varphi_0(y^t(t-\varepsilon))-\varphi_0(y^t(t))$$

and by Lemma 2 and Eqs. (2.3), (2.4) we have

$$\frac{\partial\varphi}{\partial t}(t,x)\geqslant g(y^t(t))+h(u^t(t))+(\nabla\varphi_0(y^t(t)),(y^t)'(t))$$
$$=g(y^t(t))-h^*(B^*p^t(t))+(B^*p^t(t),\nabla h^*(B^*p^t(t))$$
$$+(p^t(t),Ay^t(t)+Fy^t(t)-B\nabla h^*(B^*p^t(t)))=\delta_t \qquad (2.30)$$

because in virtue of (2.4),

$$h(u^t) + h^*(B^*p^t) = (u^t, B^*p^t) \qquad \text{in} \quad [0, t].$$

On the other hand, we have

$$\varphi(t+\varepsilon, x) - \varphi(t, x) \leqslant \int_t^{t+\varepsilon} (g(z^t) + h(v^t))\, ds$$
$$+ \varphi_0(z^t(t+\varepsilon)) - \varphi_0(z^t(t))$$

where $v^t(s) = u^t(s)$ for $s \in [0, t]$; $v^t(s) = u^t(t)$ for $s \in [t, t+\varepsilon]$ and

$$(z^t)' + Az^t + Fz^t = Bv^t \qquad \text{in} \quad [0, t+\varepsilon]$$
$$z^t(0) = x.$$

This yields

$$\frac{\partial \varphi}{\partial t}(t, x) \leqslant g(y^t(t)) + h(u^t(t)) + (\nabla \varphi_0(y^t(t)), (y^t)'(t)) = \delta_t.$$

Hence

$$\frac{\partial \varphi}{\partial t}(t, x) = \delta_t = (Ay^t(0) + Fy^t(0), p^t(0)) - h^*(B^*p^t(0)) + g(y^t(0))$$

and by Lemma 3 we conclude that

$$\frac{\partial \varphi}{\partial t}(t, x) + (Ax + Fx, \eta) + h^*(-B^*\eta) = g(x) \tag{2.31}$$

for all $\eta \in D_x^- \varphi(t, x)$.

In the general case, consider the sequence $\{\varphi_0^\varepsilon\}$

$$\varphi_0^\varepsilon(x) = \varphi_0(e^{-A\varepsilon}x), \qquad \varepsilon > 0. \tag{2.32}$$

Let φ^ε be the corresponding function defined by (1.8), i.e.,

$$\varphi^\varepsilon(t, x) = \int_0^t (g(y_\varepsilon^t) + h(u_\varepsilon^t))\, ds + \varphi_0^\varepsilon(y_\varepsilon^t(t)) \tag{2.33}$$

where

$$\begin{aligned} (y_\varepsilon^t)' + Ay_\varepsilon^t + Fy_\varepsilon^t &= Bu_\varepsilon^t \qquad \text{in} \quad [0, t] \\ y_\varepsilon^t(0) &= x. \end{aligned} \tag{2.34}$$

According to the first part of the proof, we have

$$\frac{\partial \varphi^\varepsilon}{\partial t}(t, x) + h^*(B^* p_\varepsilon^t(0)) - (Ax + Fx, p_\varepsilon^t(0)) = g(x) \qquad \text{a.e.} \quad t \in [0, T],$$
$$\varphi^\varepsilon(0, x) = \varphi_0^\varepsilon(x) \tag{2.35}$$

where p_ε^t satisfy Eqs. (2.3), (2.4), i.e.,

$$\begin{aligned} (p_\varepsilon^t)' - A^* p_\varepsilon^t - (F'(y_\varepsilon^t))^* p_\varepsilon^t &= g(y_\varepsilon^t) \qquad \text{in} \quad [0, t] \\ p_\varepsilon^t(t) &= -e^{-A\varepsilon} \nabla \varphi_0(e^{-A^*\varepsilon} y_\varepsilon^t(t)) \end{aligned} \tag{2.36}$$

$$u_\varepsilon^t = \nabla h^*(B^* p_\varepsilon^t) \qquad \text{in} \quad [0, t]. \tag{2.37}$$

By (2.33) and Assumptions 5, 6 we see that $\{u_\varepsilon^t\}$ is bounded in $L^2(0, t; U)$. Therefore on a subsequence, again denoted $\{\varepsilon\}$, we have for $\varepsilon \to 0$

$$\begin{aligned} u_\varepsilon^t &\to \tilde{u}^t \qquad && \text{weakly in } L^2(0, t; U) \\ y_\varepsilon^t &\to \tilde{y}^t && \text{strongly in } C([0, t]; H) \\ p^t(s) &\to \tilde{p}^t(s) && \text{strongly in } H \text{ for all } s \in [0, t] \end{aligned}$$

where $(\tilde{u}^t, \tilde{y}^t, \tilde{p}^t)$ satisfy Eqs. (2.2), (2.3), (2.4). Now by (2.1) we have

$$\varphi_\varepsilon(t, x) \leqslant \int_0^t (g(y^t) + h(u^t))\, ds + \varphi_0^\varepsilon(y^t(t))$$

and we infer that

$$\varphi_\varepsilon(t, x) \to \varphi(t, x) \qquad \text{for} \quad \varepsilon \to 0$$

and

$$\varphi(t, x) = \int_0^t (g(\tilde{y}^t) + h(\tilde{u}^t))\, ds + \varphi_0(\tilde{y}^t(t)).$$

Then letting $\varepsilon \to 0$ in (2.35) we obtain for every $x \in D(A)$ and a.e. $t \in [0, T]$,

$$\lim_{\varepsilon \to 0} \frac{\partial \varphi_\varepsilon}{\partial t}(t, x) + h^*(B^* \tilde{p}^t(0)) - (Ax + Fx, \tilde{p}^t(0)) = g(x).$$

Since as seen in Lemma 3, $\tilde{p}^t(0) \in -D_x^- \varphi(t, x)$ we have

$$\frac{\partial \varphi}{\partial t}(t, x) + h^*(-B^*\eta) + (Ax + Fx, \eta) = g(x) \qquad \text{a.e.} \quad t \in [0, T]$$

for some $\eta \in D_x^- \varphi(t, x)$. This completes the proof.

Proof of Theorem 2. Let (y^*, u^*) be any optimal pair in problem (1.2), (1.3). Then by the maximum principle we have (see Eq. (2.9))

$$\varphi(T, x) = \int_0^T (g(y^*(\tau)) + h(u^*(\tau)))\, d\tau + \varphi_0(y^*(T)))$$
$$= \int_0^s (g(y^*(\tau)) + h(u^*(\tau)))\, d\tau + \varphi(T-s, y^*(s)) \qquad \forall s \in [0, T].$$

Hence

$$\varphi(T-s, y^*(s)) = \int_s^T (g(y^*(\tau)) + h(u^*(\tau)))\, d\tau + \varphi_0(y^*(T))$$
$$= \inf\left\{\int_s^T (g(y) + h(u))\, d\tau + \varphi_0(y(T));\ y + Ay + Fy = Bu \quad \text{in} \quad [s, T];\ y(s) = y^*(s)\right\} \qquad \forall s \in [0, T]. \tag{2.38}$$

Then there is $p \in C([s, T]; H)$ such that (see Eqs. (2.3), (2.4))

$$\begin{aligned} &p' - A^*p - (F'(y^*))^*\, p = \nabla g(y^*) \qquad \text{in} \quad [s, T] \\ &p(T) = -\nabla\varphi_0(y^*(T)) \\ &u^*(\tau) = \nabla h^*(B^*p(\tau)) \qquad \forall \tau \in [s, T]. \end{aligned} \tag{2.39}$$

Now by Lemma 3 we have

$$-p(s) \in D_y^- \varphi(T-s, y^*(s)) \qquad \forall s \in [0, T] \tag{2.40}$$

and along with (2.39) the latter yields (1.13) as claimed.

DEFINITION ([5, 6]). The continuous function $\varphi\colon (0, T) \times H \to R$ is a viscosity solution of (1.1) provided for all $\psi \in C^1((0, T) \times H)$

if $\varphi - \psi$ attains a local maximum at $(t, x) \in (0, T) \times D(A)$ then $(\partial\psi/\partial t)(t, x) + h^*(-B^*D_x\psi(t, x)) + (Ax + Fx, D_x\psi(t, x)) \leqslant g(x)$ (2.41)

and

if $\varphi - \psi$ attains a local minimum at (t, x) then $(\partial\psi/\partial t)(t, x) + h^*(-B^*D_x\psi(t, x)) + (Ax + Fx, D_x\psi(t, x)) \geqslant g(x)$. (2.42)

PROPOSITION 1. *Under assumptions* 1–6, φ *is a viscosity solution to Eq.* (1.1).

Proof. If $\varphi-\psi$ attains a local maximum at (t, x) then we have

$$\varphi(t, x)-\psi(t, x) \geqslant \varphi(t-s, y(s))-\psi(t-s, y(s)) \quad \text{for} \quad 0 \leqslant s \leqslant t$$

where

$$\begin{aligned} y'+Ay+Fy=Bu \quad & \text{in} \quad [0, s] \\ y(0)=x. & \end{aligned} \tag{2.43}$$

This yields (see (2.9))

$$\psi(t, x)-\psi(t-s, y(s)) \leqslant \int_0^s (g(y(\tau))+h(u(\tau)))\, d\tau \qquad \forall s \in [0, t].$$

Hence for u sufficiently smooth and $x \in D(A)$ we have

$$\frac{\partial\psi}{\partial t}(t, x)-\left(\frac{\partial\psi}{\partial x}(t, x), Bu(0)-Ax-Fx\right) \leqslant g(x)+h(u(0)).$$

Finally, by the conjugacy formula (1.7) we have

$$\frac{\partial\psi}{\partial t}(t, x)+\left(Ax+Fx, \frac{\partial\psi}{\partial x}(t, x)\right)+h^*\left(-B^*\frac{\partial\psi}{\partial x}(t, x)\right) \leqslant g(x).$$

If $\varphi(t, x)-\psi(t, x)$ attains a local minimum at (t, x) then

$$\varphi(t, x)-\psi(t, x) \leqslant \varphi(t-s, y^t(s))-\psi(t-s, y^t(s)) \qquad \forall s \in [0, t]$$

where (y^t, u^t) is optimal in problem (1.8) (see (2.1), then again by (2.9) we have

$$\begin{aligned} &(t-s)^{-1}(\psi(t, x)-\psi(t-s, y^t(s))) \\ &\qquad \leqslant (t-s)^{-1}\int_0^s (g(y^t(\tau))+h(u^t(\tau)))\, d\tau. \end{aligned}$$

Hence

$$\frac{\partial\psi}{\partial t}(t, x)-\left(\frac{\partial\psi}{\partial x}(t, x), Bu^t(0)-Ax-Fx\right) \geqslant g(x)+h(u^t(0))$$

and by Eq. (2.4) we see that

$$\frac{\partial\psi}{\partial t}(t, x) + h^*\left(B^* \frac{\partial\psi}{\partial x}(t, x)\right) + \left(\frac{\partial\psi}{\partial x}(t, x), Ax + Fx\right) \geqslant g(x)$$

as claimed.

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